I. Knot Theory

A. Knots and Links

Recall a knot is the image of an embedding

$$f : S^1 \hookrightarrow \mathbb{R}^3$$

(for now f a smooth embedding)

so $K = \operatorname{im}(f)$ a knot

we say 2 knots K_0 and K_1 are isotopic if there is a smooth map

$$H : S^1 \times [0,1] \longrightarrow \mathbb{R}^3$$

such that

1) $\operatorname{im}(H|_{S^1 \times \{i\}}) = K_i \qquad i = 0,1$

2) $H|_{S^1 \times \{t\}} : S^1 \to \mathbb{R}^3$ is an embedding $\forall t \in [0,1]$

the idea is that you can smoothly deform K_0 into K_1

(i.e. if K_0 is made out of string, you can move it around to get K_1)

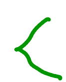
when we say 2 knots are "the same" we mean they are isotopic

knots are frequently studied via their diagrams

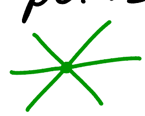
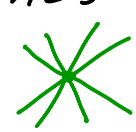
let $p : \mathbb{R}^3 \to \mathbb{R}^2 : (x,y,z) \mapsto (x,y)$ be projection

given a knot K one can show it can be isotoped (by a very small amount) such that

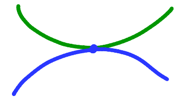
1) $p|_K$ is an immersion (that is derivative non zero)

so you can't see  ← corners → 

2) $p|_K$ has no n-tuple points for $n \geq 3$

don't see  or  . . .

2) at each douple point the two arcs of K intersect transversely ⟶ (tangent vectors of arcs a double point span $\mathbb{R}^2$)

don't see 

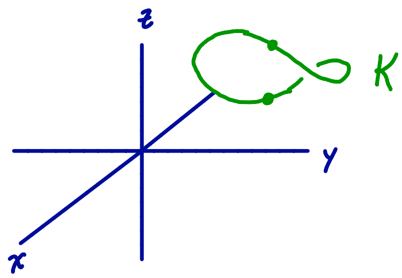
(to prove this need "jet transversality" or PL-topology, beyond this course, but hopefully believable)

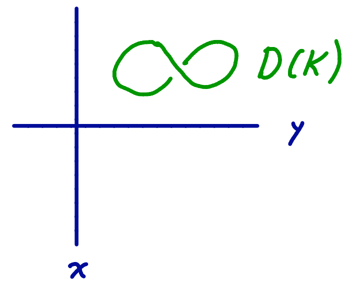
a diagram $D(K)$ of K is

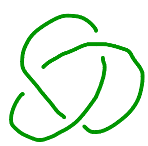
1) $p(K) \subset \mathbb{R}^2$ and

2) at each double point lable which strand goes over the other one
(i.e. which has the greater z-coordinate)

examples:

1) 



2)  ↙ trefoil

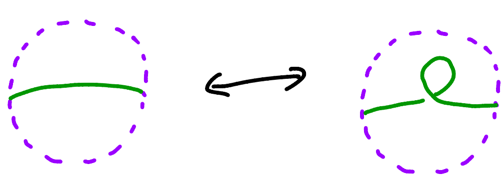
exercise: Show a knot diagram D determines a unique knot in $\mathbb{R}^3$ upto isotopy

we have an amazing theorem

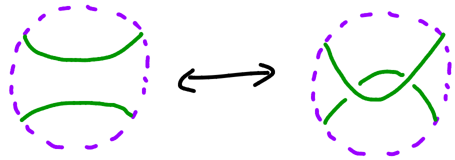
Reidemeister's $Th^{\underline{m}}$:

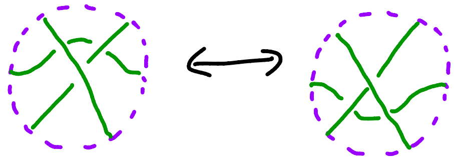
let K_0 and K_1 be knots with diagrams D_0 and D_1
Then K_0 is isotopic to K_1 $\Leftrightarrow$ D_0 is related to D_1 by a sequence of

0) deformations where crossings don't change

I) 

(this means, if you see a piece of the diagram looking like one side you can replace it with the other)

II) 

III) 

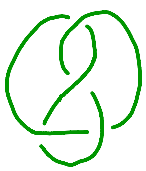
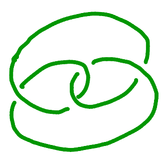
(and reflections of these about coordinate planes)

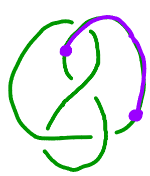
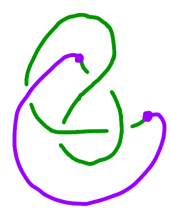
note that ($\Leftarrow$) should be clear

($\Rightarrow$) takes some work

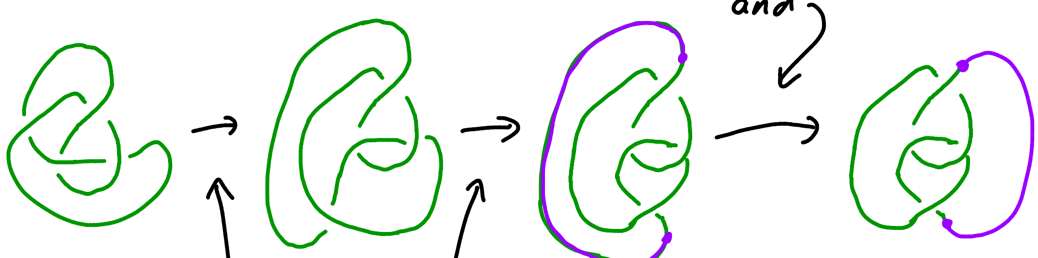
(to prove need "parametric jet transversality" or PL-topology)

example:

 and  are isotopic

to see this note  find the Reidemeister moves doing this! 

and



just push arcs around not changing crossings

A link is just a disjoint union of knots

B. Knot coloring

So how can you tell if two knots are different?

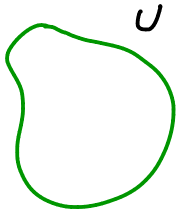
here is a very simple way

we say a knot diagram D is 3-colorable if you can color the strands of D with 3 colors so that

(a) at each crossing either all 3 colors are used or only 1 is used

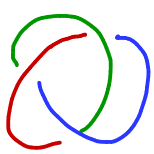
(b) at least 2 colors are used

examples:

1) 

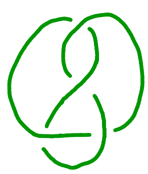
U

unknot U is not 3-colorable

2) 

T

the trefoil T is 3-colorable

exercise: 

F

show the "figure 8" knot F is not 3-colorable

Th$^{\underline{m}}$ 1:

If one diagram for a knot is 3-colorable then all diagrams are

(so being 3-colorable is a property of the knot, not just the diagram)

Remark: So from above we see the trefoil T is different from the unknot U and figure 8, F

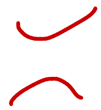
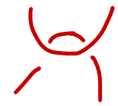
Proof:

we just need to check 3-colorability is unchanged under Reidemeister moves

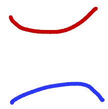
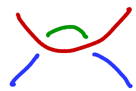
I)  $\leftrightarrow$ 

one color — must only be one color here

so ⓐ true and ⓑ true for one $\Leftrightarrow$ true for the other

II) either  $\leftrightarrow$ 

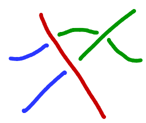
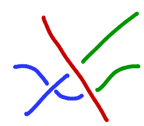
one color — one color

or  $\leftrightarrow$ 

more than one color — more than one color

so ⓐ true and ⓑ true for one $\Leftrightarrow$ true for othe

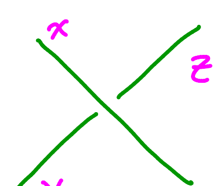
III) lots of cases

here is one  $\leftrightarrow$ 

exercise: check all other cases

more generally we say a diagram (and knot) is p-labelable for p a prime, if we can label the strands with numbers $0, 1, \ldots, p-1$ so that

ⓐ at each crossing, the overcrossing label is the mod p average of the labels of the undercrossings



$2x \equiv y + z \mod p$

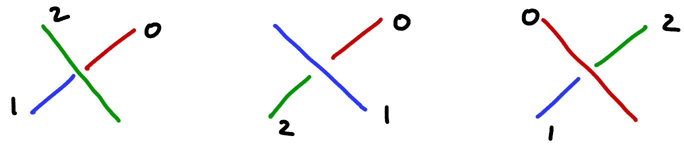
ⓑ at least 2 labels are used

exercise: Prove the analog of Th$^{\underline{m}}$1 for p-labeling

examples:

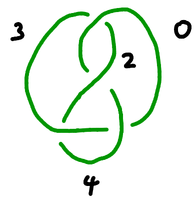
1) a 3-coloring is a 3-labeling

let red $= 0$ blue $= 1$ green $= 2$



$2\cdot 2 = 4 \equiv 0+1 \bmod 3$ $2\cdot 1 = 2 \equiv 2+0 \bmod 3$ $2\cdot 0 = 0 \equiv 1+2 \bmod 3$

2)



is a 5-labeling of F

so F is not isotopic to U

later in the course we will see how coloring/labeling is related to really cool topology!

(dihedral representations of the fundamental group of the knot complement)

C. Alexander polynomial

with a direction on each component

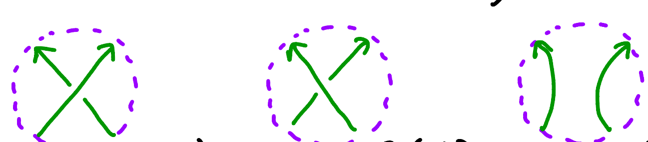
later we will prove that to any link K we can associate a (Laurent) polynomial, in the variable $t^{1/2}$, $\Delta_K(t)$ with integer coefficients (this means $\Delta_K(t)$ has the form $a_n t^{n/2} + a_{n+1} t^{n+1/2} + \dots + a_m t^{m/2}$ where $n \le m$, and a_i are integers)

(when K a knot, $\Delta_K(t)$ only has integer powers of t)

such that A) K isotopic to K', then $\Delta_K(t) = \Delta_{K'}(t)$

B) if K_+, K_-, and K_0 have diagrams related by

this is called a skein relation



$D(K_+)$ $D(K_-)$ $D(K_0)$

then $\Delta_{K_+} - \Delta_{K_-} + (t^{-1/2} - t^{1/2})\Delta_{K_0} = 0$

c) $\Delta_{\text{unknot}} = 1$

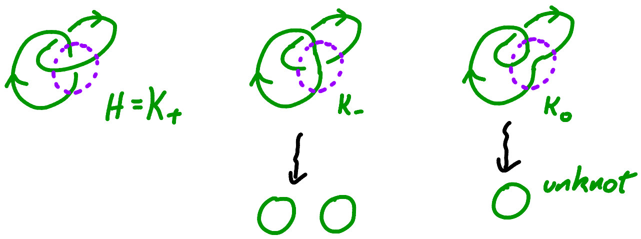
$\Delta_K(t)$ is called the (Conway normalized) Alexander polynomial of K

lemma 2:

> If K has a diagram $D(K)$ with 2 components that are separated by a line then $\Delta_K = 0$

Remark: This says Δ_K can detect something interesting about links.

example: Compute Δ_H for $H =$ 



$\Delta_{K_-} = 0$ by lemma $\qquad \Delta_{K_0} = 1$ by c)

so by B)

$$\Delta_{K_+} - \Delta_{K_-} + (t^{-1/2} - t^{1/2})\Delta_{K_0} = 0$$

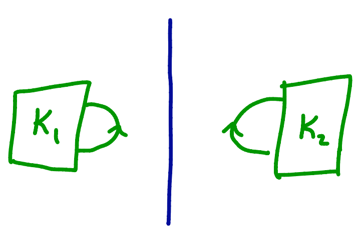
$$\Delta_H - 0 + (t^{-1/2} - t^{1/2})1 = 0$$

so $\boxed{\Delta_H(t) = t^{1/2} - t^{-1/2}}$

Remark: So we see by the lemma that the 2 componants of H can't be pulled appart!

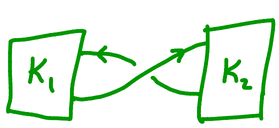
(this is "obvious", but can you come up with an easier proof? there is one but its not that much easier)

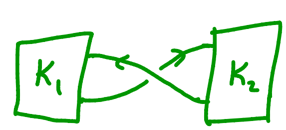
Proof of lemma:

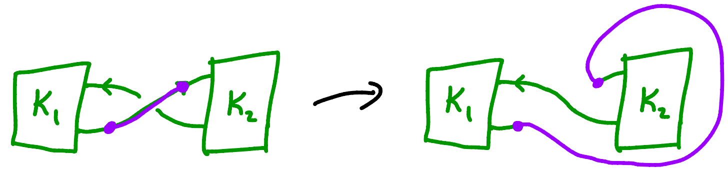
we have $K =$ 

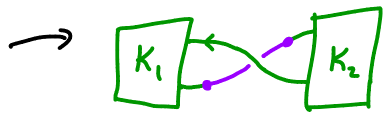
orient components so you see picture

let $D(K) = D(K_0)$

so $D(K_+) =$ 

and $D(K_-) =$ 

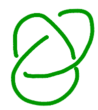
note: $D(K_+)$ 

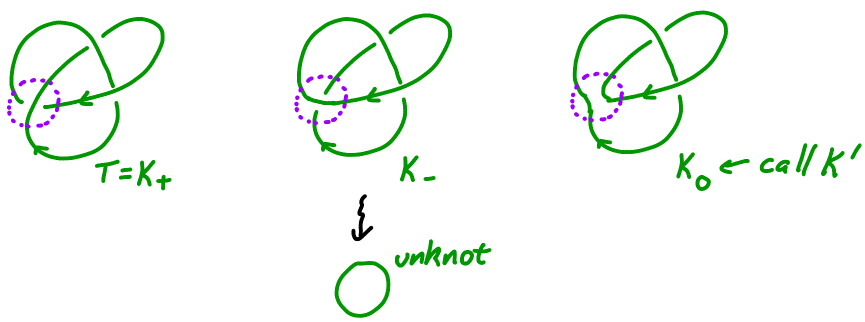
 $D(K_-)$

so by B) in definition

$$\underbrace{\Delta_{K_+} - \Delta_{K_-}}_{=0} + (t^{-1/2} - t^{1/2})\Delta_{K_0} = 0$$

so $\Delta_K = \Delta_{K_0} = 0$

example: compute Δ_K where $T =$ 



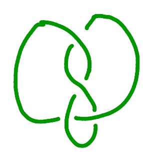
so $\boxed{\Delta_T - 1 + (t^{-1/2} - t^{1/2})\Delta_{K'} = 0}$

above we saw $\Delta_{K'} = t^{1/2} - t^{-1/2}$

so $\Delta_T = 1 + (t^{-1/2} - t^{1/2})(t^{-1/2} - t^{1/2})$

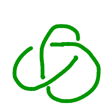
$= 1 + t^{-1} - t^0 - t^0 + t^1 = \boxed{t^{-1} - 1 + t}$

exercises:

1) Compute $\Delta_F = -t^{-1} + 3 - t$ where $F =$ 

2) Compute $\Delta_{K_{2,5}} = t^{-2} - t^{-1} + 1 - t + t^2$ where $K_{2,5} =$ 

3) Compute $\Delta_{m(T)} = t^{-1} - 1 + t$ where

$m(T) =$ 

in general, given a knot K and a diagram $D(K)$ of K we define the <u>mirror</u> of K, denoted $m(K)$, to be the knot with diagram obtained by switching all crossings of $D(K)$

4) In general, if K is a link with k components show

$$\Delta_{m(K)} = (-1)^{k-1}\Delta_K$$

Warning, this is harder than other exercises

Hint: use skein relation and induction on number of crossings

5) Show $m(F)$ is isotopic to F, where F is the figure 8 knot

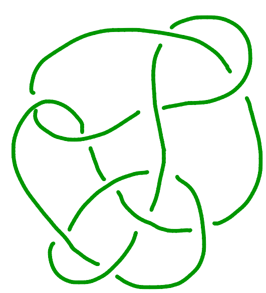
Remarks:

1) the above shows $U, T, F, K_{2,5}$ are all distinct knots but we don't know if T and $m(T)$ are distinct! (we will see they are)

2) Δ_K distinguishes, up to mirroring, all (prime) knots with ≤ 8 crossings (in a diagram) and most with 9 crossings

3) there are many knots with $\Delta_K = 1$ (so can't be distinguished from unknot)

e.g.



4) It is not hard to show (you should try!) that A), B), C) uniquely specify Δ_K if Δ_K is well-defined

Note: we have not show Δ_K is well-defined. This can be done using Reidemeister moves, but it is <u>much</u> easier and more enlightning to prove this with ideas we develop later in the course.

Δ_K can also be used to understand things about a knot!

For example

1) We will see that for every knot K there is a surface Σ in $\mathbb{R}^3$ with boundary K

e.g.

this surface has more "holes" than

later we will make precise the idea that Σ has 2 holes and Σ' has 4 holes

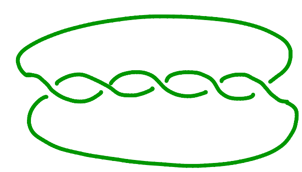
important and hard question: What is the smallest number of holes in a surface with boundary K?

we will see

$$\boxed{\#\text{holes} \geq \text{breadth } \Delta_K}$$

biggest degree in Δ_K - smallest degree

example: from above any surface Σ with boundary $K_{2,5}$ must have at least 4 holes!



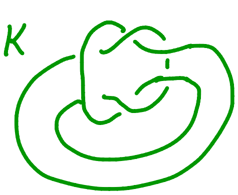
$$\Delta_{K_{2,5}} = t^{-2} - t^{-1} + 1 - t + t^2$$

we see this later

2) Given a knot $K \subset \mathbb{R}^3 \subset S^3$, there is a surface $\Sigma \subset B^4$ with boundary K (we will try to visualize these later)

Very hard question: What is the minimal number of holes for such a surface? Can it be a disk?

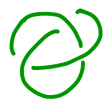
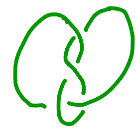
example: Show K



does not bound a disk in $\mathbb{R}^3$ (that is, K is not the unknot) but K does bound a disk in B^4.

later we might see

$\boxed{\text{if } K \text{ bounds a disk in } B^4 \text{ (many } K \text{ do) then there is a polynomial } f \text{ with integer coefficients such that } \Delta_K(t) = f(t)\,f(t^{-1})}$

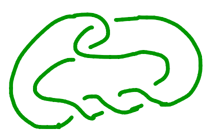
example: from above  and  do not bound disks in B^4.

3) Is there a function $f:(S^3-K) \to S^1$ such that $df \neq 0$?
Such a knot is called fibered and it is very helpful to know if a knot is fibered.

later we might see

$\boxed{\text{if } K \text{ is fibered, then the coefficient of the highest order term in } \Delta_K \text{ is } \pm 1}$

example: you can check $\Delta_{Tw} = 2t^{-1} - 3 + 2t$

where $Tw =$ 

so Tw is not fibered

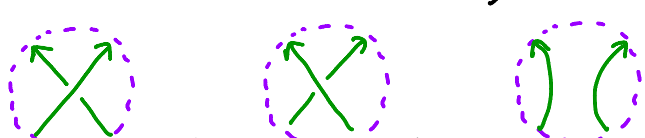
D. Jones polynomial

later we will prove that to any link K (with a direction on each component) we can associate a (Laurent) polynomial, in the variable $t^{1/2}$, $V_K(t)$ with integer coefficients

(when K a knot, $\Delta_K(t)$ only has integer powers of t)

such that A) K isotopic to K', then $V_K(t) = V_{K'}(t)$

B) if K_+, K_-, and K_0 have diagrams related by

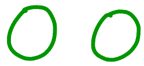


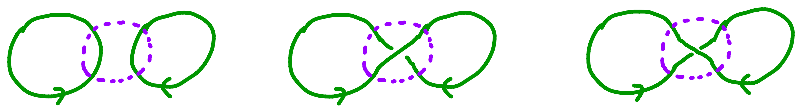
$D(K_+)$ $D(K_-)$ $D(K_0)$

then $t^{-1}V_{K_+} - tV_{K_-} - (t^{1/2} - t^{-1/2})V_{K_0} = 0$

c) $V_{\text{unknot}} = 1$

examples:

1) let $O_2 =$  2 component unlink



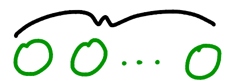
$O_2 = K_0$ $\quad$ $K_+ = U$ $\quad$ $K_- = U$

so $\quad t^{-1}V_{K_+} - tV_{K_-} - (t^{1/2} - t^{-1/2})V_{O_2} = 0$

and $\quad V_{O_2} = \dfrac{t^{-1} - t}{t^{1/2} - t^{-1/2}} = -t^{1/2} - t^{-1/2}$

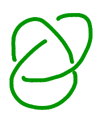
exercise: let O_k be the k-component unlink

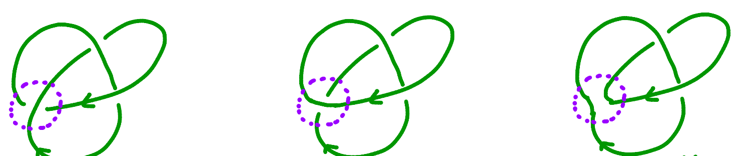
k times



then show $\quad V_{O_k} = (-t^{1/2} - t^{-1/2})^{k-1}$

Hint: induction

2) let $T =$  compute V_T

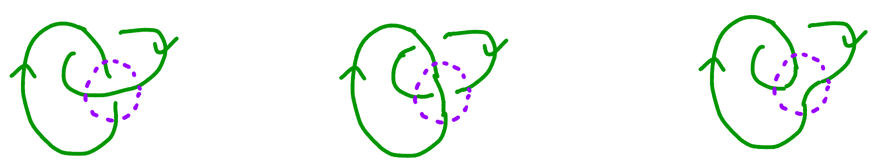


$T = K_+$ $\quad$ K_- $\quad$ $K_0 \leftarrow$ call K'

K_- $\rightsquigarrow$ unknot

so $\quad t^{-1}V_T - tV_U - (t^{1/2} - t^{-1/2})V_{K'} = 0$

let's compute $V_{K'}$



$K' = K_+$ $\quad$ K_- $\quad$ K_0

$K_- \rightsquigarrow O_2$ $\quad$ $K_0 \rightsquigarrow$ unknot so $t^{-1}V_{K'} - tV_{O_2} - (t^{1/2}-t^{-1/2})V_U = 0$

and $t^{-1}V_{K'} = t(-t^{1/2}-t^{-1/2}) + (t^{1/2}-t^{-1/2})$

so $V_{K'} = -t^{5/2} - \cancel{t^{3/2}} + \cancel{t^{3/2}} - t^{1/2}$

$= -t^{5/2} - t^{1/2}$

now for V_T: $t^{-1}V_T - tV_U - (t^{1/2}-t^{-1/2})V_{K'} = 0$

so $V_T = t\left(t + (t^{1/2}-t^{-1/2})(-t^{5/2}-t^{1/2})\right)$

$= t(\cancel{t} - t^3 + t^2 - \cancel{t} + 1)$

$$\boxed{V_T = -t^4 + t^3 + t}$$

<u>exercise</u>:

1) recall the mirror $m(T)$ of T is 

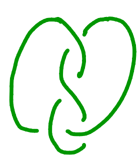
compute $V_{m(T)} = t^{-1} + t^{-3} - t^{-4}$

so $m(T)$ and T are <u>not</u> isotopic!

(the Alexander polynomial and coloring can't see this!)

2) More generally show $V_{m(K)}(t) = V_K(t^{-1})$ for any knot.

<u>Hint</u>: maybe wait till we have another definition of V_K

3) for $F =$  compute

$V_F = t^{-2} - t^{-1} + 1 + t + t^2$

note $V_F(t^{-1}) = V_F(t)$ which is good since $m(F)$ isotopic to F

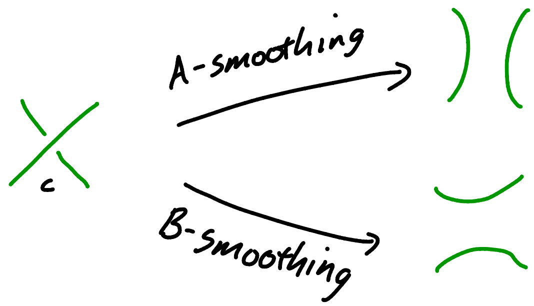
<u>Much studied unsolved problem</u>:

Is there a nontrivial knot K such that $V_K = 1$?

Unlike the Alexander polynomial, the Jones polynomial does not seem to "see" interesting topological things, but it can still tell us interesting things

Before we get to that, let's give another definition of V_K from which we can see that it is well defined.

given a diagram D for a link and a crossing c in D there are 2 natural other diagrams you can construct



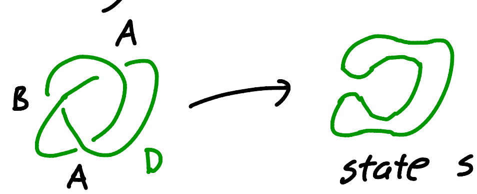
(moving on lower strand turn <u>right</u> at crossing)

(turn <u>left</u>)

denote by $|D|$ the number of components of the link associated to D

a state s of a diagram is a choice of A or B smoothing at each crossing

<u>example</u>:



for a state s of D let

$\alpha(s)$ = number of A-smoothings of s

$\beta(s)$ = number of B-smoothings of s

$|s|$ = number of components of s

define the <u>bracket</u> of D by

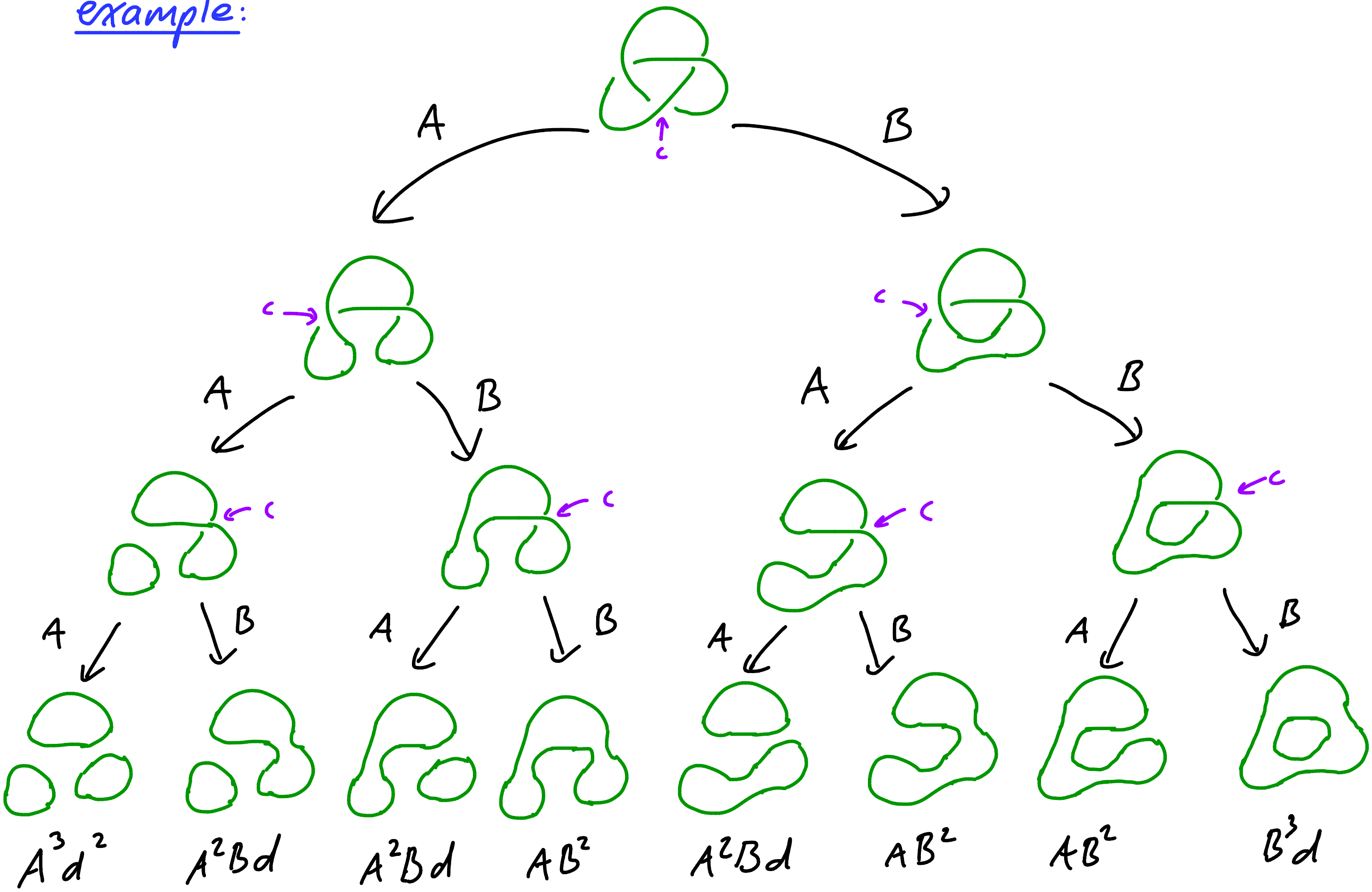
$$\langle D \rangle = \sum_{\substack{\text{all states}\\ s \text{ of } D}} A^{\alpha(s)} B^{\beta(s)} d^{|s|-1}$$

where A, B, and d are formal variables so $\langle \cdot \rangle : \{\text{link diagrams}\} \to \mathbb{Z}[A, B, d]$

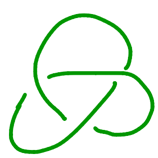
is a well-defined function

set of integer valued polynomials in the variables A, B, d

example:



so

$\langle$  $\rangle = A^3d^2 + (3A^2B + B^3)d + 3AB^2$

note: $\langle\ \rangle$ satisfies

unlink with k components (only has empty state)

1) $\langle O_k \rangle = d^{k-1}$

2) $\langle$  $\rangle = A \langle)(\rangle + B \langle$  $\rangle$

exercise: 1) if this is not clear to you then prove it!
(maybe look back at last example)

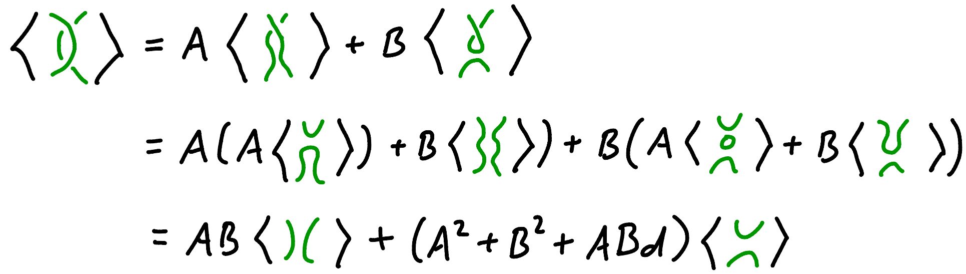
2) also show $\langle D \amalg O \rangle = d \langle D \rangle$

lemma 3:

> if we set $B = A^{-1}$ and $d = -(A^2 + A^{-2})$ then $\langle\cdot\rangle$ is invariant under Reidemeister moves II and III

Proof:

II)



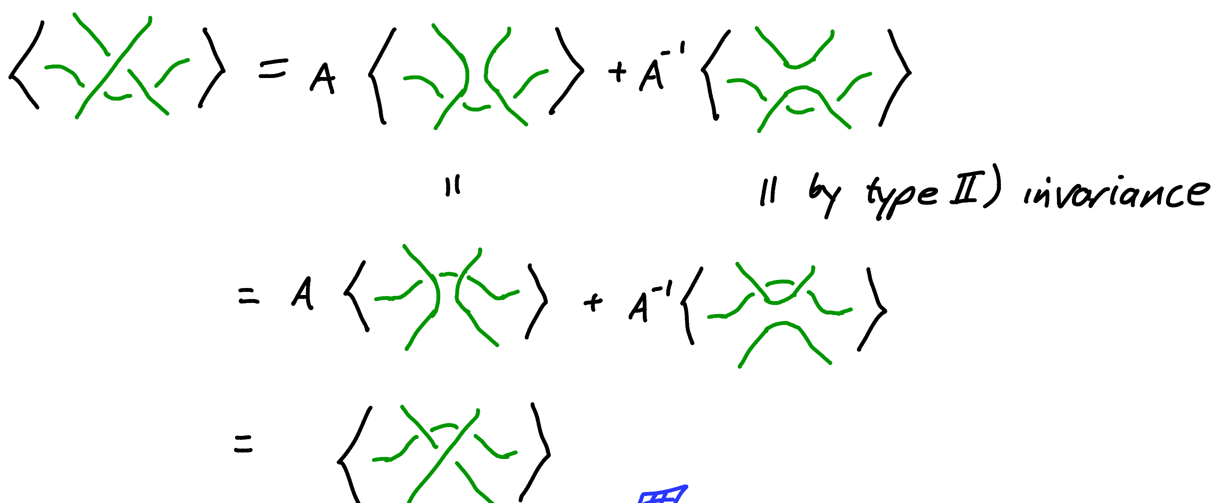
so for 

we need $AB = 1$ so $B = A^{-1}$

and $A^2 + B^2 + ABd = 0$ so $d = -(A^2 + A^{-2})$

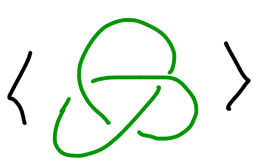
(you can check other type II) moves give same rel^{n})

III)



With $B = A^{-1}$ and $d = -(A^{-2} + A^2)$ we get $\langle K \rangle$ a polynomial in the variable A. This is called the Kauffman bracket of K

example:



$$= A^3 d^2 + (3A^2B + B^3)d + 3AB^2$$

$$= A^3(-A^{-2} - A^2)^2 + (3A^2A^{-1} + A^{-3})(-A^{-2} - A^2) + 3AA^{-2}$$

$$= A^3(\cancel{A^{-4}} + 2 + A^4) - \cancel{3A^{-1}} - 3A^3 - A^{-5} - \cancel{A^{-1}} + \cancel{3A^{-1}}$$

$$= A^7 - A^3 - A^{-5}$$

What about Reidemeister type I) move?

$$\langle\ \rangle = A\langle\ \rangle + B\langle\ \rangle$$
$$= (Ad+B)\langle\ \rangle$$
$$= (A(-(A^{-2}+A^{2}))+A^{-1})\langle\ \rangle$$
$$= -A^{3}\langle\ \rangle$$

$$\langle\ \rangle = A\langle\ \rangle + B\langle\ \rangle$$
$$= (A+Bd)\langle\ \rangle$$
$$= (A + A^{-1}(-(A^{-2}+A^{2})))\langle\ \rangle$$
$$= (-A^{-3})\langle\ \rangle$$

to fix this we define the writhe of D as follows:

for an oriented diagram D set

$\varepsilon(\ ) = 1$ and $\varepsilon(\ ) = -1$

right handed crossing — left handed crossing

the <u>writhe</u> of D is $\quad w(D) = \sum_{\substack{\text{crossings} \\ c}} \varepsilon(c)$

note:

1) $w(\ ) = w(\ )$ — and similarly for the other type II) moves

2) $w(\ ) = w(\ )$ — and similarly for other type III) moves

3) $w(\ ) = w(\ ) - 1$

$w(\ ) = w(\ ) + 1$

So if we set $F_D(A) = (-A)^{-3w(D)}\langle D\rangle$ for an oriented diagram D

then $F_D(A)$ is invariant under all Reidemeister moves!

so $F_D(A)$ is an invariant of the link associated to D

$$F : \{\text{oriented links}\} \longrightarrow \mathbb{Z}[A]$$

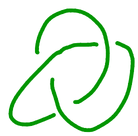
example:

for $m(T)=$  we have $\omega(m(T)) = -3$

so $F_{m(T)}(A) = -A^9\langle T\rangle = -A^9(A^7 - A^3 - A^{-5})$

$= -A^{16} + A^{12} + A^4$

exercise:

1) if $T=$ , then show $F_T(A) = -A^{-16} + A^{-12} + A^{-4}$

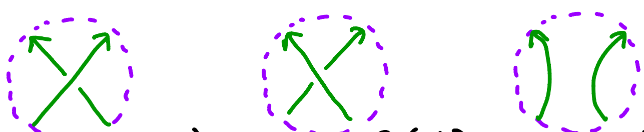
2) Show $F_{m(K)}(A) = F_K(A^{-1})$

3) if $\bar{K}$ is K with the opposite orientation then $F_{\bar{K}}(A) = F_K(A)$

4) If O_k is the k component unlink then $F_{O_k} = (-A^2 - A^{-2})^{k-1}$

Th$^{\underline{m}}$ 4:

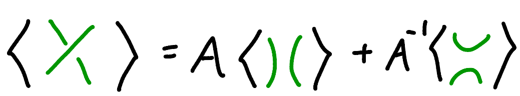
if K_+, K_-, and K_0 have diagrams related by

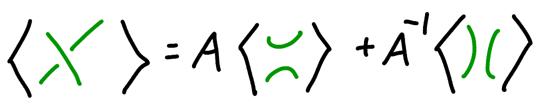


$D(K_+)$ $D(K_-)$ $D(K_0)$

then $$A^4 F_{K_+} - A^{-4} F_{K_-} + (A^{-2} - A^2) F_{K_0} = 0$$

Proof:





$\therefore \; A\langle K_+\rangle - A^{-1}\langle K_-\rangle = (A^2 - A^{-2})\langle K_0\rangle$

$\omega(K_\pm) = \omega(K_0) \pm 1$

$$\therefore \quad F_{K_+} = (-A)^{-3\omega_0-3} \langle K_+ \rangle \Rightarrow (-A)^{-3\omega_0} \langle K_+ \rangle = (-A)^{3} F_{K_+}$$

$$F_{K_-} = (-A)^{-3\omega_0+3} \langle K_- \rangle \Rightarrow (-A)^{-3\omega_0} \langle K_- \rangle = (-A)^{-3} F_{K_-}$$

$$F_{K_0} = (-A)^{-3\omega_0} \langle K_0 \rangle$$

and we have

$$A^4 F_{K_+} - A^{-4} F_{K_-} - (A^{-2} - A^2) F_{K_0} = 0$$

note: if we set $V_K(t) = F_L(t^{-1/4})$ then we see V_K satisfies

A) V_K an invariant of isotopy class of K

B) $t^{-1} V_{K_+} - t V_{K_-} - (t^{1/2} - t^{-1/2}) V_{K_0} = 0$

C) $V_{unknot} = 1$

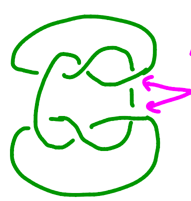
i.e. V_K is the Jones polynomial!

and now we know it is well-defined!

E. Alternating Links

a knot diagram D is called alternating if over and under crossings alternate as you traverse the knot

alternating

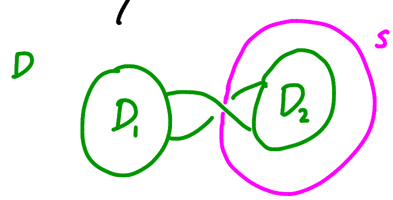


not alternating

a link is alternating if it has an alternating diagram

an alternating diagram is called reduced if there is no embedded circle in $\mathbb{R}^2$ intersecting the diagrame transversely one time at a crossing

not reduced!



can remove this crossing $\Rightarrow$